Journal of Engineering Mathematics (2024) 148:15
https://doi.org/10.1007/s10665-024-10401-y

Approximate frequency analysis of isotropic shear beams using initial value method

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Received: 12 May 2024 / Accepted: 17 September 2024
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Abstract
This study investigates the vibration of isotropic shear beams (ISBs) using the unified shear deformation theory for various shear stress distribution functions. The governing differential equations and boundary conditions are obtained using the principle of virtual work. Using the initial values method, the first five frequencies for various end conditions were calculated with four digits. Frequencies for different shear stress distributions can be easily determined using the approximate transfer matrices provided in this study. To verify the accuracy of the obtained results, we compared the frequencies with the findings in the existing literature.

Keywords Isotropic shear beams · Initial values method · Transfer matrix · Approximate transfer matrix · Vibrration of shear beams · Matricant

1 Introduction

Functionally graded materials (FGMs) exhibit unique properties due to their tailored composition, making them advantageous for various engineering applications. These

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Published online: 08 October 2024

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materials are characterized by a gradual change in composition, microstructure, and properties over their volume [1–4]. FGMs offer superior mechanical, thermal, and electrical properties compared to traditional homogeneous materials [5–8]. One notable application of FGMs is in the fabrication of functionally graded beams [9–12]. These beams possess varying material properties along their length, enabling them to efficiently withstand complex loading conditions. The gradient in material composition allows for enhanced structural performance, including improved stiffness, strength, and fatigue resistance. Additionally, FGM beams can exhibit better thermal stability and resistance to thermal stress compared to conventional homogeneous beams. Numerous studies have investigated the mechanical behavior and performance of functionally graded materials and beams. References such as [1–4] provide comprehensive insights into the design, fabrication, and analysis of FGMs and FGM-based structures. These references cover topics ranging from theoretical modeling and numerical simulations to experimental characterization and practical applications, highlighting the versatility and potential of functionally graded materials in engineering and technology.

Cowper [13, 14] has confirmed the validity of Timoshenko beam theory in predicting the fundamental frequency for transverse vibrations of simply supported beams using a plane stress exact elasticity solution. However, due to the shortcomings of the elementary theory of beam and first-order shear deformation theory, several higher-order shear deformation theories have been developed for more accurate static and dynamic analysis of beams. These theories can be found in numerous literature sources [15–26], [27]. Reddy [1] has proposed a well-known third-order shear deformation theory to analyze plates with moderate thickness nonlinearly.

Karama [28] and Soldatos [29] proposed different shear deformation theories, one using exponential functions and the other using hyperbolic functions. Three theories on trigonometric shear deformation are presented by Touratier [30], Vlasov and Leont'ev [31], and Stein [32] for thick beams. However, it is important to note that these theories fail to satisfy the shear stress-free boundary conditions at the top and bottom surfaces of the beam. Ghugal and Shipmi [33] and Ghugal [34] have developed a trigonometric shear deformation theory that satisfies the shear stress-free condition at the top and bottom surfaces of the beam. Recently, it has become important to examine the mechanical behavior of beams produced with FGMs with the help of combined unified shear deformation theory. This article will support studies in this direction. The study aimed to calculate the vibration frequencies for different end conditions using the unified shear deformation theory with varying shear stress distribution functions. The frequencies were obtained using the initial values method. The energy principle was utilized to derive the basic equations and boundary conditions. Studying the vibration of functionally graded beams using the initial value method will interest researchers working in this field. The essence of the initial value method is to find the transfer matrix of a system of differential equations. The transfer matrix is unique for a system of differential equations. After seeing this matrix, the solution of the problem for each end condition can be found by solving the algebraic system of equations (it is much easier to solve a system of algebraic equations than a system of differential equations). However, it may not always be possible to find the exact transfer matrix. In such a case, the problem can be easily solved by using approximate transfer matrices. In this

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study, the problem is solved with an approximate transfer matrix. The derivation of this matrix is explained in detail. The frequencies for other beam theories can be easily found with the help of the transfer matrix given in this study.

2 Vibration of ISB

The displacement field of the present unified shear deformation theory is given as follows:

$$\hat{u}(\bar{x},z,t) = -z\frac{\partial \hat{w}(\bar{x},t)}{\partial \bar{x}} + f(z)\hat{\varphi}(\bar{x},t) \tag{1}$$

If the new variables are defined as follows,

$$\hat{w}(\bar{x},t) = e^{i\bar{\omega}t}\overline{w}(\bar{x}),\ \hat{\varphi}(\bar{x},t) = e^{i\bar{\omega}t}\overline{\varphi}(\bar{x}) \tag{2}$$

Eq. (1) can be written in the following form:

$$\hat{u}(\bar{x},z,t) = -ze^{i\bar{\omega}t}\bar{w}' + f(z)e^{i\bar{\omega}t}\bar{\varphi} \tag{3}$$

Normal strain and the transverse shear strain are

$$\epsilon_{\bar{x}} = \frac{\partial \hat{u}}{\partial \bar{x}} = -ze^{i\bar{\omega}t}\bar{w}^{(2)} + e^{i\bar{\omega}t}f(z)\bar{\varphi}',\ \gamma_{z\bar{x}} = \frac{\partial \hat{u}}{\partial z} + \frac{\partial \hat{w}}{d\bar{x}} = f'(z)e^{i\bar{\omega}t}\bar{\varphi} \tag{4}$$

Axial stress and shear stress are

$$\sigma_{\bar{x}} = E\epsilon_{\bar{x}},\ \tau_{z\bar{x}} = G\gamma_{z\bar{x}}, \tag{5}$$

where E is elasticity modulus and G is shear modulus. Variation of virtual work is

$$\delta U = \int_0^L \int_{-\frac{h}{2}}^{+\frac{h}{2}} (\sigma_{\bar{x}}\delta\epsilon_{\bar{x}} + \tau_{z\bar{x}}\delta\gamma_{z\bar{x}})\,\mathrm{d}z\mathrm{d}\bar{x} + \rho\int_0^L \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(\frac{\partial^2\hat{u}}{dt^2}\delta\hat{u} + \frac{\partial^2\hat{w}}{dt^2}\delta\hat{w}\right)\mathrm{d}z\mathrm{d}\bar{x}, \tag{6}$$

where h is the height of the beam and L is the length of the beam. Using the principle of work ($\delta U = 0$), the basic equations and boundary conditions are obtained as follows. Basic equations are

$$A_0\bar{w}^{(4)} - A_1\bar{\varphi}^{(3)} + \frac{\rho\bar{\omega}^2}{E}A_0\bar{w}^{(2)} - \frac{\rho\bar{\omega}^2}{E}A_1\bar{\varphi}' - \rho\bar{\omega}^2 h\bar{w} = 0 \tag{7}$$

$$A_1\bar{w}^{(3)} - A_2\bar{\varphi}^{(2)} + A_3\bar{\varphi} + \frac{\rho\bar{\omega}^2}{E}A_1\bar{w}' - \frac{\rho\bar{\omega}^2}{E}A_2\bar{\varphi} = 0, \tag{8}$$

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where

$$A_0 = E\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2\,\mathrm{d}z,\ A_1 = E\int_{-\frac{h}{2}}^{\frac{h}{2}} zf(z)\,\mathrm{d}z,$$
$$A_2 = E\int_{-\frac{h}{2}}^{\frac{h}{2}} f(z)^2\,\mathrm{d}z,\ A_3 = G\int_{-\frac{h}{2}}^{\frac{h}{2}} f'(z)^2\,\mathrm{d}z \tag{9}$$

Boundary conditions are

$$(-A_0\bar{w}^{(3)} + A_1\bar{\varphi}^{(2)} - \frac{\rho\bar{\omega}^2}{E}A_0\bar{w}' + \frac{\rho\bar{\omega}^2}{E}A_1\bar{\varphi})\,\delta w = 0 \tag{10}$$
$$(A_0\bar{w}^{(2)} - A_1\bar{\varphi}')\,\delta\bar{w}' = 0 \tag{11}$$
$$(-A_1\bar{w}^{(2)} + A_2\bar{\varphi}')\,\delta\bar{\varphi} = 0 \tag{12}$$

These basic equations and boundary conditions are also obtained in reference [35]. The new variables are defined as follows:

$$\bar{\alpha}_1 = A_0\bar{w}^{(2)} - A_1\bar{\varphi}',\ \bar{\alpha}_2 = \bar{\alpha}_1' = A_0\bar{w}^{(3)} - A_1\bar{\varphi}^{(2)} \tag{13}$$
$$\bar{\beta}_1 = A_1\bar{w}' - A_2\bar{\varphi},\ \bar{\beta}_2 = \bar{\beta}_1' = A_1\bar{w}^{(2)} - A_2\bar{\varphi}' \tag{14}$$

Using Eqs. 13 and 14, $\bar{w}'$ and $\bar{\varphi}'$ are obtained as follows:

$$\bar{w}' = \frac{1}{A_1}\bar{\beta}_1 + \frac{A_2}{A_1}\bar{\varphi},\ \bar{\varphi}' = \frac{A_0}{{A_1}^2 - A_0A_2}\bar{\beta}_2 - \frac{A_1}{{A_1}^2 - A_0A_2}\bar{\alpha}_1 \tag{15}$$

Equations 7 and 8 can be written in terms of new variables as follows:

$$\bar{\alpha}_2' + \frac{\rho\bar{\omega}^2}{E}\bar{\alpha}_1 - \rho\bar{\omega}^2h\bar{w} = 0 \tag{16}$$
$$\bar{\beta}_2' + A_3\bar{\varphi} + \frac{\rho\bar{\omega}^2}{E}\bar{\beta}_1 = 0 \tag{17}$$

The boundary conditions given by Eqs. 10–12 can be written in terms of new variables as follows:

$$\left(-\bar{\alpha}_2 - \frac{A_0\rho\bar{\omega}^2}{A_1E}\bar{\beta}_1 + \left(\frac{A_1\rho\bar{\omega}^2}{E} - \frac{A_0A_2\rho\bar{\omega}^2}{A_1E}\right)\bar{\varphi}\right)\delta\bar{w} = 0 \tag{18}$$
$$\bar{\alpha}_1\,\delta(\frac{1}{A_1}\bar{\beta}_1 + \frac{A_2}{A_1}\bar{\varphi}) = 0 \tag{19}$$
$$\bar{\beta}_2\,\delta\bar{\varphi} = 0 \tag{20}$$

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3 Dimensionless parameters

Dimensionless variables are defined as follows:

$$\alpha_1 = \frac{\bar{\alpha}_1}{EL^2},\ \alpha_2 = \frac{\bar{\alpha}_2}{EL},\ \beta_1 = \frac{\bar{\beta}_1}{EL^3},\ \beta_2 = \frac{\bar{\beta}_2}{EL^2}$$
$$w = \frac{\bar{w}}{L},\ x = \frac{\bar{x}}{L}, S = \frac{h}{L},\ \omega = \bar{\omega}\left(\frac{L^2}{h}\right)\sqrt{\frac{\rho}{E}},\ \varphi = \bar{\varphi} \tag{21}$$

Basic Eqs. (16, 17) can be written in terms of dimensionless variables as follows:

$$\alpha_2{}' + S^2\omega^2\alpha_1 - S^3\omega^2 w = 0 \tag{22}$$
$$\beta_2{}' + B_3\varphi + S^2\omega^2\beta_1 = 0, \tag{23}$$

where

$$B_3 = \frac{S}{2(1+\nu)h}\int_{-\frac{h}{2}}^{\frac{h}{2}} f'(z)^2\,\mathrm{d}z \tag{24}$$

μ is the Poisson ratio. Boundary conditions (18–20) can be written in terms of dimensionless variables as follows:

$$\left(-\alpha_2 - \frac{A_0}{A_1}\omega^2 S^2\beta_1 + \frac{\omega^2 S^5}{h^3}\left(\frac{A_1}{E} - \frac{A_0 A_2}{A_1 E}\right)\varphi\right)\delta w = 0 \tag{25}$$
$$\alpha_1\delta w' = \alpha_1\,\delta\left(\frac{Eh^3}{S^3 A_1}\beta_1 + \frac{A_2}{A_1}\varphi\right) = 0 \tag{26}$$
$$\beta_2\,\delta\varphi = 0 \tag{27}$$

Derivatives of dimensionless variables are listed.

$$\alpha_1{}' = \alpha_2,\ \alpha_2{}' = -S^2\omega^2\alpha_1 + S^3\omega^2 w,\ \beta_1{}' = \beta_2$$
$$\beta_2{}' = -B_3\varphi - S^2\omega^2\beta_1,\ w' = \frac{Eh^3}{A_1 S^3}\beta_1 + \frac{A_2}{A_1}\varphi$$
$$\varphi' = \frac{A_0 Eh^3}{({A_1}^2 - A_0 A_2)S^3}\beta_2 - \frac{A_1 Eh^3}{({A_1}^2 - A_0 A_2)S^3}\alpha_1 \tag{28}$$

The system in Eq. 28 can be written in matrix form as follows:

$$\frac{d\boldsymbol{y}(x)}{\mathrm{d}x} = \boldsymbol{A}\,\boldsymbol{y}(x), \tag{29}$$

where

$$\boldsymbol{y} = (\alpha_1, \alpha_2, \beta_1, \beta_2, w, \varphi)^T. \tag{30}$$

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Here, T is the transpose operator. The matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -S^2\omega^2 & 0 & 0 & 0 & S^3\omega^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -S^2\omega^2 & 0 & 0 & -B_3 \\ 0 & 0 & \frac{Eh^3}{A_1S^3} & 0 & 0 & \frac{A_2}{A_1} \\ \frac{A_1Eh^3}{(A_0A_2-A_1^2)S^3} & 0 & 0 & \frac{A_0Eh^3}{(A_1^2-A_0A_2)S^3} & 0 & 0 \end{pmatrix} \tag{31}$$

4 Methods used in this study

The initial values method is employed for frequency calculations. This method first requires determining the exact transfer matrix of the problem's mathematical model (See Eq. 28). However, in cases where obtaining the exact transfer matrix proves challenging due to its complexity or the presence of lengthy terms, the approximate transfer matrix (matricant) derived from Eq. 33 offers a valuable alternative.

a) Exact transfer matrix

There is the following relationship between the solution of the system Eq. 28, the exact transfer matrix (ETM), and the initial values:

$$\boldsymbol{y}(x) = \boldsymbol{ETM}\, \boldsymbol{y}(0). \tag{32}$$

Here, y(x) is the column matrix consisting of the unknowns of the problem.

b) Matricant (MCNT)

Matricant is the approximate transfer matrix and is found as follows [36]:

$$\begin{aligned} \boldsymbol{MTCNT}(x, x_1) = \boldsymbol{I} &+ \int_{x_1}^{x} \boldsymbol{A}(\tau)\,\mathrm{d}\tau + \int_{x_1}^{x} \boldsymbol{A}(\tau) \int_{x_1}^{\tau} \boldsymbol{A}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &+ \int_{x_1}^{x} \boldsymbol{A}(\tau) \int_{x_1}^{\tau} \boldsymbol{A}(\sigma) \int_{x_1}^{\sigma} \boldsymbol{A}(\beta)\,\mathrm{d}\beta\mathrm{d}\sigma\,\mathrm{d}\tau + \ldots \end{aligned} \tag{33}$$

Matricant satisfies the following relation:

$$\begin{aligned} \boldsymbol{MCNT}(0, x) \;=\; & \boldsymbol{MCNT}(0, x_1)\boldsymbol{MCNT}(x_1, x_2)\boldsymbol{MCNT}(x_2, x_3) \\ & \ldots \boldsymbol{MCNT}(x_{n-1}, x) \end{aligned} \tag{34}$$

It is possible to improve the value of the transfer matrix at a point x, which is found with a very small number of terms, with the help of Eq. 34

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4.1 Elements of approximate transfer matrices

The transfer matrices for the system of differential equations given by Eq. 28 are provided for two and three terms.

Two terms matricant is

$$
\begin{aligned}
TM_{1,1} &= 1-\frac{1}{2}S^2\left(x-x_1\right)^2\omega^2,\ TM_{1,2}=x-x_1,\ TM_{1,3}=0\\
TM_{1,4} &= 0,\ TM_{1,5}=\frac{1}{2}S^3\left(x-x_1\right)^2\omega^2,\ TM_{1,6}=0\\
TM_{2,1} &= S^2\left(x_1-x\right)\omega^2,\ TM_{2,2}=1-\frac{1}{2}S^2\left(x-x_1\right)^2\omega^2\\
TM_{2,3} &= \frac{Eh^3\left(x-x_1\right)^2\omega^2}{2A_1},\ TM_{2,4}=0\\
TM_{2,5} &= S^3\left(x-x_1\right)\omega^2,\ TM_{2,6}=\frac{A_2S^3\left(x-x_1\right)^2\omega^2}{2A_1}\\
TM_{3,1} &= 0,\ TM_{3,2}=0,\ TM_{3,3}=1-\frac{1}{2}S^2\left(x-x_1\right)^2\omega^2\\
TM_{3,4} &= x-x_1,\ TM_{3,5}=0,\ TM_{3,6}=-\frac{1}{2}B_3\left(x-x_1\right)^2\\
TM_{4,1} &= \frac{A_1B_3Eh^3\left(x-x_1\right)^2}{2\left(A_1^2-A_0A_2\right)S^3},\ TM_{4,2}=0,\ TM_{4,3}=S^2\left(x_1-x\right)\omega^2\\
TM_{4,4} &= \frac{1}{2}\left(\frac{A_0B_3Eh^3\left(x-x_1\right)^2}{\left(A_0A_2-A_1^2\right)S^3}-S^2\left(x-x_1\right)^2\omega^2+2\right)\\
TM_{4,5} &= 0,\ TM_{4,6}=B_3\left(x_1-x\right)\\
TM_{5,1} &= -\frac{A_2Eh^3\left(x-x_1\right)^2}{2\left(A_1^2-A_0A_2\right)S^3},\ TM_{5,2}=0,\ TM_{5,3}=\frac{Eh^3\left(x-x_1\right)}{A_1S^3}\\
TM_{5,4} &= \frac{A_1Eh^3\left(x-x_1\right)^2}{2\left(A_1^2-A_0A_2\right)S^3},\ TM_{5,5}=1,\ TM_{5,6}=\frac{A_2\left(x-x_1\right)}{A_1}\\
TM_{6,1} &= \frac{A_1Eh^3\left(x_1-x\right)}{\left(A_1^2-A_0A_2\right)S^3},\ TM_{6,2}=-\frac{A_1Eh^3\left(x-x_1\right)^2}{2\left(A_1^2-A_0A_2\right)S^3}\\
TM_{6,3} &= -\frac{A_0Eh^3\left(x-x_1\right)^2\omega^2}{2\left(A_1^2-A_0A_2\right)S},\ TM_{6,4}=\frac{A_0Eh^3\left(x-x_1\right)}{\left(A_1^2-A_0A_2\right)S^3}\\
TM_{6,5} &= 0,\ TM_{6,6}=1-\frac{A_0B_3Eh^3\left(x-x_1\right)^2}{2\left(A_1^2-A_0A_2\right)S^3}
\end{aligned}
\tag{35}
$$

Three terms matricant is

$$
TM_{1,1} = 1-\frac{1}{2}S^2\left(x-x_1\right)^2\omega^2,\ TM_{1,2}=-\frac{1}{6}S^2\left(x-x_1\right)^3\omega^2+x-x_1
$$

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$$
\begin{aligned}
TM_{1,3} &= \frac{Eh^3 (x-x_1)^3 \omega^2}{6A_1},\ TM_{1,4} = 0 \\
TM_{1,5} &= \frac{1}{2} S^3 (x-x_1)^2 \omega^2,\ TM_{1,6} = \frac{A_2 S^3 (x-x_1)^3 \omega^2}{6A_1} \\
TM_{2,1} &= \frac{1}{2}\omega^2 \left(\frac{A_2 Eh^3 (x-x_1)^3}{3A_0A_2 - 3A_1^2} + \frac{1}{3} S^4 (x-x_1)^3 \omega^2 + 2S^2 (x_1 - x) \right) \\
TM_{2,2} &= 1 - \frac{1}{2} S^2 (x-x_1)^2 \omega^2,\ TM_{2,3} = \frac{h^3 (x-x_1)^2 \omega^2}{2A_1} \\
TM_{2,4} &= \frac{A_1 Eh^3 (x-x_1)^3 \omega^2}{6\left(A_1^2 - A_0A_2\right)} \\
TM_{2,5} &= S^3 (x-x_1) \omega^2 - \frac{1}{6} S^5 (x-x_1)^3 \omega^4 \\
TM_{2,6} &= \frac{A_2 S^3 (x-x_1)^2 \omega^2}{2A_1} \\
TM_{3,1} &= \frac{A_1 B_3 Eh^3 (x-x_1)^3}{6\left(A_1^2 - A_0A_2\right) S^3},\ TM_{3,2} = 0 \\
TM_{3,3} &= 1 - \frac{1}{2} S^2 (x-x_1)^2 \omega^2 \\
TM_{3,4} &= \frac{1}{6} (x-x_1) \left(\frac{A_0 B_3 Eh^3 (x-x_1)^2}{\left(A_0A_2 - A_1^2\right) S^3} - S^2 (x-x_1)^2 \omega^2 + 6 \right) \\
TM_{3,5} &= 0,\ TM_{3,6} = -\frac{1}{2} B_3 (x-x_1)^2 \\
TM_{4,1} &= \frac{A_1 B_3 Eh^3 (x-x_1)^2}{2\left(A_1^2 - A_0A_2\right) S^3},\ TM_{4,2} = \frac{A_1 B_3 Eh^3 (x-x_1)^3}{6\left(A_1^2 - A_0A_2\right) S^3} \\
TM_{4,3} &= (x-x_1) \omega^2 \Big(A_0 B_3 \mathrm{EE} h^3 (x-x_1)^2 \\
&\quad + \left(A_1^2 - A_0A_2\right) S^3 \left(S^2 (x-x_1)^2 \omega^2 - 6 \right) \Big) \\
&\quad /6\left(A_1^2 - A_0A_2\right) S \\
TM_{4,4} &= \frac{1}{2} \left(\frac{A_0 B_3 Eh^3 (x-x_1)^2}{\left(A_0A_2 - A_1^2\right) S^3} - S^2 (x-x_1)^2 \omega^2 + 2 \right) \\
TM_{4,5} &= 0 \\
TM_{4,6} &= B_3 (x-x_1) \Big(A_0 B_3 Eh^3 (x-x_1)^2 \\
&\quad + \left(A_1^2 - A_0A_2\right) S^3 \left(S^2 (x-x_1)^2 \omega^2 - 6 \right) \Big) \\
&\quad /6\left(A_1^2 - A_0A_2\right) S^3 \\
TM_{5,1} &= -\frac{A_2 Eh^3 (x-x_1)^2}{2\left(A_1^2 - A_0A_2\right) S^3},\ TM_{5,2} = -\frac{A_2 Eh^3 (x-x_1)^3}{6\left(A_1^2 - A_0A_2\right) S^3}
\end{aligned}
$$

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$$
\begin{aligned}
TM_{5,3} &= -\frac{Eh^3(x-x_1)\left(A_1^2\left(S^2(x-x_1)^2\omega^2-6\right)+6A_0A_2\right)}{6\left(A_1^3-A_0A_1A_2\right)S^3}\\
TM_{5,4} &= \frac{A_1Eh^3(x-x_1)^2}{2\left(A_1^2-A_0A_2\right)S^3},\ TM_{5,5}=1\\
TM_{5,6} &= \frac{A_2(x-x_1)}{A_1}-\frac{A_1B_3Eh^3(x-x_1)^3}{6\left(A_1^2-A_0A_2\right)S^3}\\
TM_{6,1} &= A_1Eh^3(x-x_1)\left(A_0B_3Eh^3(x-x_1)^2\right.\\
&\quad\left.+\left(A_1^2-A_0A_2\right)S^3\left(S^2(x-x_1)^2\omega^2-6\right)\right)\\
&\quad/6\left(A_1^2-A_0A_2\right)^2S^6\\
TM_{6,2} &= -\frac{A_1Eh^3(x-x_1)^2}{2\left(A_1^2-A_0A_2\right)S^3},\ TM_{6,3}=-\frac{A_0Eh^3(x-x_1)^2\omega^2}{2\left(A_1^2-A_0A_2\right)S}\\
TM_{6,4} &= A_0(-E)h^3(x-x_1)\left(A_0B_3Eh^3(x-x_1)^2\right.\\
&\quad\left.+\left(A_1^2-A_0A_2\right)S^3\left(S^2(x-x_1)^2\omega^2-6\right)\right)\\
&\quad/6\left(A_1^2-A_0A_2\right)^2S^6\\
TM_{6,5} &= \frac{A_1Eh^3(x-x_1)^3\omega^2}{6A_0A_2-6A_1^2},\ TM_{6,6}=1-\frac{A_0B_3\mathrm{EE}h^3(x-x_1)^2}{2\left(A_1^2-A_0A_2\right)S^3}
\end{aligned}
\tag{36}
$$

5 Calculation of frequencies using different theories and methods

The expressions suggested by scientists for the $f(z)$ function seen in Eq. (1) are given as follows [15, 20, 29, 30, 37–39]

$$
\begin{aligned}
f(z) &= \frac{1}{2}z\left(\frac{h^2}{4}-\frac{z^2}{3}\right);\ f(z)=\frac{5}{4}z\left(1-\frac{4z^2}{3h^2}\right);\\
f(z) &= z\left(1-\frac{4z^2}{3h^2}\right);\ f(z)=\frac{h\sin\left(\frac{\pi z}{h}\right)}{\pi};\\
f(z) &= z\cosh\left(\frac{1}{2}\right)-h\sinh\left(\frac{z}{h}\right);\ f(z)=ze^{-\frac{2z^2}{h^2}};\\
f(z) &= \frac{3}{2}\pi\left(h\tanh\left(\frac{z}{h}\right)-z\mathrm{sech}^2\left(\frac{1}{2}\right)\right);
\end{aligned}
\tag{37}
$$

5.1 Frequencies of simple beam

The boundary conditions are obtained with the help of Eqs. 25–27 as follows:

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Table 1 For simple beams, frequencies found with the help of an exact transfer matrix ($h = 12, \mu = 3/10$, $S = 1/4$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	2.5958	8.5691	15.7932	23.4347	31.1935
Kruszewski	2.5958	8.5691	15.7932	23.4346	31.1934
Reddy	2.5958	8.5690	15.7932	23.4347	31.2390
Touratier	2.5961	8.5733	15.8112	23.4829	31.3392
Karama	2.5970	8.5823	15.8428	23.5574	31.1948
Soldatos	2.5958	8.5689	15.7921	23.4315	31.1972
Akavci	2.5959	8.5716	15.8046	23.4661	31.1751

Table 2 For simple beams, frequencies found with the help of an exact transfer matrix ($h = 12, \mu = 3/10$, $S = 1/10$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	2.8023	10.7093	22.5663	37.1637	53.5567
Kruszewski	2.8024	10.7093	22.5663	37.1637	53.5567
Reddy	2.8023	10.7093	22.5663	37.1637	53.5567
Touratier	2.8024	10.7101	22.5702	37.1754	53.5834
Karama	2.8025	10.7124	22.5802	37.2021	53.6391
Soldatos	2.8023	10.7093	22.5662	37.1634	53.5557
Akavci	2.8023	10.7097	22.5685	37.1704	53.5726

$$\begin{aligned} \alpha_1(0) = 0,\ \beta_2(0) = 0,\ w(0) = 0 \\ \alpha_1(1) = 0,\ \beta_2(1) = 0,\ w(1) = 0 \end{aligned} \tag{38}$$

If Eq. 32 is taken into consideration with the given boundary conditions, the following determinant must be zero.

$$\begin{vmatrix} \mathrm{TM}_{1,2}(1) & \mathrm{TM}_{1,3}(1) & \mathrm{TM}_{1,6}(1) \\ \mathrm{TM}_{4,2}(1) & \mathrm{TM}_{4,3}(1) & \mathrm{TM}_{4,6}(1) \\ \mathrm{TM}_{5,2}(1) & \mathrm{TM}_{5,3}(1) & \mathrm{TM}_{5,6}(1) \end{vmatrix} = 0 \tag{39}$$

a) Exact transfer matrix
The exact transfer matrix is not provided due to the large number of terms in its elements.

b) Method of Matricant (The number of intervals is the number n in Eq. 34)

5.2 The frequencies for a beam with both ends fixed

The boundary conditions are obtained with the help of Eqs. 25–27 as follows:

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Table 3 Frequencies found by the matricant method for simple beams with 2 terms and 48 intervals ($h = 12, \mu = 3/10, S = 1/4$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	2.5924	8.5307	15.6529	23.0995	30.5936
Kruszewski	2.5924	8.5307	15.6529	23.0995	30.5936
Reddy	2.5924	8.5307	15.6529	23.0995	30.5936
Touratier	2.5928	8.5349	15.6706	23.1459	30.6886
Karama	2.5936	8.5437	15.7016	23.2181	30.8248
Soldatos	2.5924	8.5305	15.6519	23.0964	30.5869
Akavci	2.5926	8.5332	15.6641	23.1297	30.6566

Table 4 Frequencies found by the matricant method for simple beams with 3 terms and 16 intervals ($h = 12, \mu = 3/10, S = 1/4$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	2.5956	8.5585	15.7078	23.0854	31.1935
Kruszewski	2.5956	8.5585	15.7078	23.0854	31.1935
Reddy	2.5955	8.5585	15.7078	23.0854	31.1935
Touratier	2.5956	8.5585	15.7078	23.0854	31.1935
Karama	2.5956	8.5585	15.7078	23.0854	31.1935
Soldatos	2.5956	8.5583	15.7068	23.0824	30.1972
Akavci	2.5958	8.5610	15.7190	23.1156	31.1751

Table 5 Frequencies found by the matricant method for simple beams with 2 terms and 48 intervals ($h = 12, \mu = 3/10, S = 1/10$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	2.7986	10.6556	22.3295	36.5256	52.2369
Kruszewski	2.7984	10.6525	22.3160	36.4886	52.1591
Reddy	2.7984	10.6525	22.3160	36.4886	52.1591
Touratier	2.7984	10.6533	22.3198	36.4997	52.1841
Karama	2.7986	10.6555	22.3295	36.5256	52.2370
Soldatos	2.7984	10.6524	22.3159	36.4882	52.1581
Akavci	2.7984	10.6529	22.3181	36.4949	52.1740

$$\begin{aligned} \beta_1(0) = 0,\ w(0) = 0,\ \varphi(0) = 0 \\ \beta_1(1) = 0,\ w(1) = 0,\ \varphi(1) = 0 \end{aligned} \tag{40}$$

($\varphi(0) = 0, w'(0) \rightarrow \beta_1(0) = 0$). If Eq. 32 is taken into consideration with the given boundary conditions, the following determinant must be zero.

$$\begin{vmatrix} \mathrm{TM}_{3,1}(1) & \mathrm{TM}_{3,2}(1) & \mathrm{TM}_{3,4}(1) \\ \mathrm{TM}_{5,1}(1) & \mathrm{TM}_{5,2}(1) & \mathrm{TM}_{5,4}(1) \\ \mathrm{TM}_{6,1}(1) & \mathrm{TM}_{6,2}(1) & \mathrm{TM}_{6,4}(1) \end{vmatrix} = 0 \tag{41}$$

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Table 6 Frequencies found by the matricant method for simple beams with 3 terms and 16 intervals ($h = 12, \mu = 3/10, S = 1/10$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	2.8020	10.6936	22.4138	36.4604	51.4318
Kruszewski	2.8021	10.6936	22.4138	36.4604	51.4318
Reddy	2.8021	10.6936	22.4138	36.4604	51.4318
Touratier	2.8021	10.6944	22.4177	36.4716	51.4560
Karama	2.8023	10.6967	22.4275	36.4973	51.5074
Soldatos	2.8020	10.6936	22.4137	36.4601	51.4308
Akavci	2.8021	10.6941	22.4159	36.4668	51.4462

Table 7 Frequencies obtained using the exact transfer matrix for a beam with both ends fixed ($h = 12, \mu = 3/10, S = 1/4$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	4.8059	10.6517	17.5340	24.8320	32.4598
Kruszewski	4.8059	10.6517	17.5340	24.8320	32.4598
Reddy	4.8059	10.6517	17.5340	24.8320	32.4598
Touratier	4.8137	10.6855	17.6114	24.9688	32.6730
Karama	4.8247	10.7298	17.7096	25.1394	32.9350
Soldatos	4.8053	10.6492	17.5280	24.8211	32.4427
Akavci	4.8110	10.6745	17.5856	24.9252	32.6055

a) Exact transfer matrix
b) Method of matricant

5.3 Beam with left end fixed and right end free

In this case, three of the initial values are zero.

$$\beta_1(0) = 0, w(0) = 0, \varphi(0) = 0, \tag{42}$$

The end conditions are (see Eqs. 25–27)

$$\left(-\alpha_2(1) - \frac{A_0}{A_1}\omega^2 S^2 \beta_1(1) + \frac{\omega^2 S^5}{h^3}\left(\frac{A_1}{E} - \frac{A_0 A_2}{A_1 E}\right)\varphi(1)\right) = 0,$$
$$\alpha_1(1) = 0,\ \beta_2(1) = 0 \tag{43}$$

If Eq. 32 is taken into consideration with the given boundary conditions, the following determinant must be zero. The elements of determinant are

$$det_{1,1} = \frac{S^2\omega^2\left(\left(A_1^2 - A_0 A_2\right) S^3 \mathrm{TM}_{61} - A_0 \mathrm{E} h^3 \mathrm{TM}_{31}\right)}{A_1 \mathrm{E} h^3} - \mathrm{TM}_{21}$$

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Table 8 Frequencies obtained using the exact transfer matrix for a beam with both ends fixed ($h = 12, \mu = 3/10, S = 1/10$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	6.0610	15.5611	28.1877	42.9049	59.0575
Kruszewski	6.0610	15.5611	28.1877	42.9049	59.0575
Reddy	6.0610	15.5611	28.1877	42.9049	59.0575
Touratier	6.0622	15.5687	28.2107	42.9550	59.1487
Karama	6.0645	15.5814	28.2471	43.0321	59.2832
Soldatos	6.0609	15.5607	28.1862	42.9016	59.0512
Akavci	6.0618	15.5659	28.2026	42.9379	59.1176

Table 9 Frequencies obtained using the matricant method for a beam with both ends fixed with 2 terms and 48 intervals ($h = 12, \mu = 3/10, S = 1/4$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	4.7959	10.5961	17.3682	24.4643	31.7762
Kruszewski	4.7959	10.5961	17.3682	24.4643	31.7762
Reddy	4.7959	10.5960	17.3682	24.4643	31.7761
Touratier	4.8035	10.6292	17.4433	24.5953	31.9775
Karama	4.8146	10.6729	17.5389	24.7592	32.2261
Soldatos	4.7953	10.5936	17.3624	24.4539	31.7601
Akavci	4.8010	10.6184	17.4192	24.5536	31.9136

Table 10 Frequencies obtained using the matricant method for a beam with both ends fixed with 3 terms and 48 intervals ($h = 12, \mu = 3/10, S = 1/4$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	4.8059	10.6515	17.5325	24.8262	32.4433
Kruszewski	4.8059	10.6515	17.5325	24.8262	32.4431
Reddy	4.8059	10.6515	17.5325	24.8262	32.4431
Touratier	4.8136	10.6852	17.6099	24.9629	32.6559
Karama	4.8247	10.7296	17.7080	25.1333	32.9175
Soldatos	4.8053	10.6489	17.5264	24.8153	32.4261
Akavci	4.8110	10.6742	17.5850	24.9133	32.5886

$$
\begin{aligned}
det_{1,2} &= \frac{S^2\omega^2\left(\left(A_1^2 - A_0A_2\right)S^3\mathrm{TM}_{62} - A_0\mathrm{E}h^3\mathrm{TM}_{32}\right)}{A_1\mathrm{E}h^3} - \mathrm{TM}_{22} \\
det_{1,3} &= \frac{S^2\omega^2\left(\left(A_1^2 - A_0A_2\right)S^3\mathrm{TM}_{64} - A_0\mathrm{E}h^3\mathrm{TM}_{34}\right)}{A_1\mathrm{E}h^3} - \mathrm{TM}_{24} \\
det_{2,1} &= TM_{1,1},\ det_{2,2} = TM_{1,2},\ det_{2,2} = TM_{1,4} \\
det_{3,1} &= TM_{4,1},\ det_{4,2} = TM_{1,2},\ det_{3,3} = TM_{4,4}
\end{aligned} \tag{44}
$$

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Table 11 Frequencies obtained using the matricant method for a beam with both ends fixed with 2 terms and 100 intervals ($h = 12, \mu = 3/10, S = 1/10$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	6.0570	15.5341	28.0990	42.6964	58.6530
Kruszewski	6.0570	15.5341	28.0990	42.6964	58.6530
Reddy	6.0571	15.5341	28.0990	42.6964	58.6530
Touratier	6.0583	15.5416	28.1218	42.7461	58.7422
Karama	6.0606	15.5542	28.1579	42.8218	58.8743
Soldatos	6.0569	15.5337	28.0976	42.6931	58.6468
Akavci	6.0578	15.5390	28.1138	42.7289	58.7118

Table 12 Frequencies obtained using the matricant method for a beam with both ends fixed with 3 terms and 100 intervals ($h = 12, \mu = 3/10, S = 1/10$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	6.0610	15.5611	28.1875	42.9041	59.0552
Kruszewski	6.0610	15.5611	28.1875	42.9041	59.0552
Reddy	6.0610	15.5611	28.1875	42.9041	59.0552
Touratier	6.0622	15.5686	28.2105	42.9546	59.1464
Karama	6.0645	15.5813	28.2469	43.0313	59.2808
Soldatos	6.0609	15.5606	28.1860	42.9007	59.0489
Akavci	6.0618	15.5659	28.2024	42.9371	59.1152

Table 13 Frequencies obtained using the exact transfer matrix for a beam with left end fixed and right end free ($h = 12, \mu = 3/10, S = 1/4$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	0.9692	4.9343	11.3688	18.3984	25.6918
Kruszewski	0.9692	4.9343	11.3688	18.3984	25.6918
Reddy	0.9692	4.9343	11.3688	18.3984	25.6918
Touratier	0.9694	4.9394	11.3910	18.4505	25.7835
Karama	0.9696	4.9469	11.4217	18.5206	25.9053
Soldatos	0.9692	4.9339	11.3672	18.3945	25.6848
Akavci	0.9693	4.9377	11.3837	18.4334	25.7535

6 Conclusions

In this study, we have used the initial value method to calculate the first five frequencies for various end conditions with an accuracy of four digits. By utilizing the approximate transfer matrices provided in Eqs. 35, and 36, the frequencies for different shear stress distributions (f(z) functions) can be easily determined. To solve a problem using the initial values method, it is crucial to obtain the transfer matrix of

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Table 14 Frequencies obtained using the matricant method for a beam with left end fixed and right end free with 2 terms and 50 intervals ($h = 12, \mu = 3/10, S = 1/4$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	0.9690	4.9229	11.3083	18.2260	25.3397
Kruszewski	0.9690	4.9229	11.3083	18.2260	25.3397
Reddy	0.9690	4.9229	11.3083	18.2260	25.3397
Touratier	0.9691	4.9280	11.3301	18.2768	25.4284
Karama	0.9693	4.9355	11.3603	18.3453	25.5468
Soldatos	0.9690	4.9226	11.3067	18.2223	25.3330
Akavci	0.9691	4.9263	11.3229	18.2601	25.3994

Table 15 Frequencies obtained using the matricant method for a beam with left end fixed and right end free with 3 terms and 50 intervals ($h = 12, \mu = 3/10, S = 1/4$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	0.9692	4.9343	11.3685	18.2260	18.3968
Kruszewski	0.9692	4.9343	11.3685	18.2260	18.3968
Reddy	0.9692	4.9343	11.3685	18.2260	18.3968
Touratier	0.9693	4.9394	11.3908	18.4489	25.7781
Karama	0.9696	4.9469	11.4214	18.5190	25.8999
Soldatos	0.9692	4.9339	11.3669	18.3930	25.6795
Akavci	0.9693	4.9376	11.3834	18.4318	25.7481

Table 16 Frequencies obtained using the exact transfer matrix for a beam with left end fixed and right end free ($h = 12, \mu = 3/10, S = 1/10$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	1.0071	6.0391	15.8989	28.8771	44.0224
Kruszewski	1.0071	6.0391	15.8989	28.8771	44.0224
Reddy	1.0071	6.0391	15.8989	28.8771	44.0224
Touratier	1.0071	6.0398	15.9036	28.8921	44.0566
Karama	1.0771	6.0413	15.9121	28.9180	44.1133
Soldatos	1.0771	6.0390	15.8986	28.8763	44.0203
Akavci	1.0071	6.0395	15.9018	28.8866	44.0443

the system of equations. This matrix is unique and calculated for the homogeneous system of equations. Once the transfer matrix is found, the frequencies for all types of end conditions can be easily determined with the help of lower-order determinants. This study employed 3×3 determinants to identify all relevant frequencies. While the exact transport matrix for the problem is not provided due to its complexity (each element containing numerous terms), the corresponding frequencies are presented in tables. To provide an approximation, Eqs. 35 and 36 depict transport matrices derived

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Table 17 Frequencies obtained using the matricant method for a beam with left end fixed and right end free with 2 terms and 50 intervals ($h = 12, \mu = 3/10, S = 1/10$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	1.0070	6.0348	15.8703	28.7819	43.7979
Kruszewski	1.0070	6.0348	15.8703	28.7819	43.7979
Reddy	1.0070	6.0348	15.8703	28.7819	43.7979
Touratier	1.0070	6.0355	15.8749	28.7968	43.8317
Karama	1.0070	6.0370	15.8834	28.8225	43.8876
Soldatos	1.0070	6.0347	15.8700	28.7810	43.7958
Akavci	1.0070	6.0352	15.8732	28.7913	48.8195

Table 18 Frequencies obtained using the matricant method for a beam with left end fixed and right end free with 3 terms and 50 intervals ($h = 12, \mu = 3/10, S = 1/10$)

Frequencies	ω_1	ω_2	ω_3	ω_4	ω_5
Ambartsumyan	1.0071	6.0391	15.8988	28.8768	44.0215
Kruszewski	1.0071	6.0391	15.8988	28.8768	44.0215
Reddy	1.0071	6.0391	15.8988	28.8768	44.0215
Touratier	1.0071	6.0398	15.9035	28.8919	44.0558
Karama	1.0071	6.0413	15.9120	28.9178	44.1125
Soldatos	1.0071	6.0391	15.8986	28.8760	44.0194
Akavci	1.0071	6.0395	15.9018	28.8864	44.0434

from Eq. 33 by including only 2 and 3 terms, respectively. Equation 34 enabled a more precise calculation of the matricant's value at its endpoint ($x = 1$). Examining the tables revealed that increasing the number of intervals in Eq. 34 yielded frequencies progressively closer to those obtained using the exact transport matrix. The matrix introduced in Eq. 35 is particularly advantageous due to its simplicity. It allows for straightforward calculation of frequencies using Eq. 34 and the matrix itself. The frequencies presented in Sect. 5.1 are consistent with the values reported in the literature (See [35]). Across various shear stress distributions (f(z) functions) proposed by researchers, the calculated frequency values exhibit remarkable agreement at low frequencies. However, these values diverge slightly at higher frequencies (See Sect. 5). Any discrepancies between the measured frequencies of an FGM beam in a laboratory setting and the frequencies predicted here would necessitate the development of a new shear stress distribution, represented by the $f(z)$ function. The proposed method enables the calculation of frequencies in an efficient manner. This can be done by using the matrices supplied in Eqs. 35 or 36 along with Eq. 34 for all $f(z)$. This facilitates a straightforward comparison between the obtained frequencies and the laboratory results, enabling a quick assessment of their compatibility.

Acknowledgements The work reported here is supported by the Alexander von Humboldt Foundation.

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